Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2007

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines the important features of graphs of a wide variety of functions, including conic sections	3, 5	
Applies appropriate algebraic techniques to complex numbers and polynomials	2, 4	
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	1, 6	
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form	7, 8	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15	/15	/15	/15	/15	/15	/15	/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \cos^5 x \sin x \; dx$. **2**

(b) Find $\int \frac{2x}{\sqrt{x^2-4}}\, dx$ using the substitution $u = x^2 - 4$. **3**

(c) (i) Express $\frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)}$ as a sum of partial fractions. **2**

(ii) Hence show that $\int_0^3 \frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)} dx = \tan^{-1}\left(\frac{3}{11}\right)$. **4**

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x}$. **4**

Question 2. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 1 + i$ and $w = 1 - 2i$. Find in the form $x + iy$,

(i) $z\overline{w}$ **1**

(ii) $3z + iw$ **1**

(iii) $\dfrac{w}{z}$ **1**

(b) Let $\beta = -1 + i$

(i) Express β in modulus-argument form. **2**

(ii) Express β^4 in modulus-argument form. **1**

(iii) Hence evaluate β^{20} **1**

(c) (i) Sketch, on the same Argand diagram, the locus specified by, **4**

1. $|z - 9| = |z + 1|$

2. $|z - 2 + i| = 2$

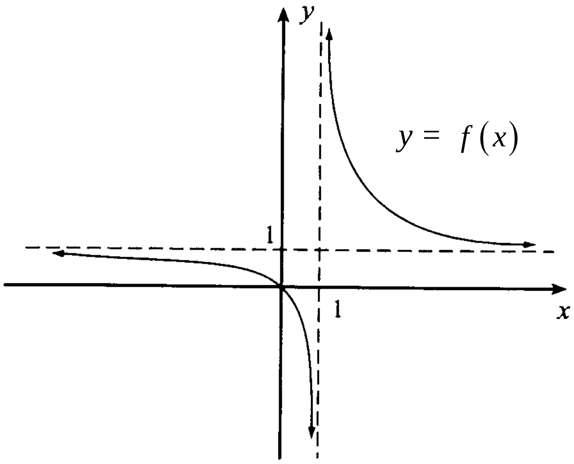
(ii) Hence write down all the values of z which satisfy simultaneously **1**

$$|z - 9| = |z + 1| \quad \text{and} \quad |z - 2 + i| = 2$$

(d) Prove $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ and interpret this result geometrically. **3**

Question 3. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram bellows shows the graph of $y = f(x)$



Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(x-1)-1$ **2**

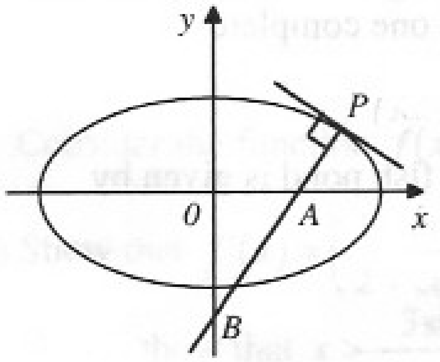
(ii) $y = \left|f(x)\right|$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y = \log_e(f(x))$ **2**

(Question 3 continues over)

(b) $P\left(a\cos\theta, b\sin\theta\right)$, where $0<\theta<\frac{\pi}{2}$, is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a > b > 0$.



The normal at *P* cuts the *x* axis at *A* and the *y* axis at *B*.

(i) Show that the normal at *P* has the equation **2**

$$ax\sin\theta - by\cos\theta = \left(a^2-b^2\right)\sin\theta\cos\theta$$

(ii) Show that triangle *OAB* has areas $\dfrac{\left(a^2-b^2\right)^2\sin\theta\cos\theta}{2ab}$ **2**

(iii) Find the maximum area of the triangle *OAB* and the coordinates of *P* when this maximum occurs. **3**

Question 4. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that α, β and γ are the roots to the equation $x^3 - x^2 + 5x - 3 = 0$, find the equation whose roots are αβ, αγ and βγ **3**

(b) Let α be the complex root of the polynomial $z^7 = 1$ with the smallest possible argument.

Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\phi = \alpha^3 + \alpha^5 + \alpha^6$

(i) Explain why $\alpha^7 = 1$ and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$ **2**

(ii) Show $\theta + \phi = -1$ and $\theta\phi = 2$ **3**

Hence write a quadratic equation whose roots are θ and ϕ

(iii) Show that $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $\phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$ **2**

(iv) Write α in modulus argument form and show **2**

$$\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2} \qquad \text{and} \qquad \sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$$

(c) The polynomial $P(z)$ is defined by $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$. **3**

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of real quadratic factors.

Question 5. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the curve given by $5y - xy = x^2 - x - 2$

(i) Show that the curve has stationary points at 5± 3√2 **2**

(ii) Explain why the curve approaches that of $y = -x - 4$ as x → ± ∞ **2**

(b) For the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$, find

(i) The eccentricity. **1**

(ii) The coordinates of the foci. **1**

(iii) The equations of the directrices. **1**

(iv) The equations of the asymptotes. **1**

(v) Sketch the hyperbola indicating the foci, the directrices and the asymptotes. **1**

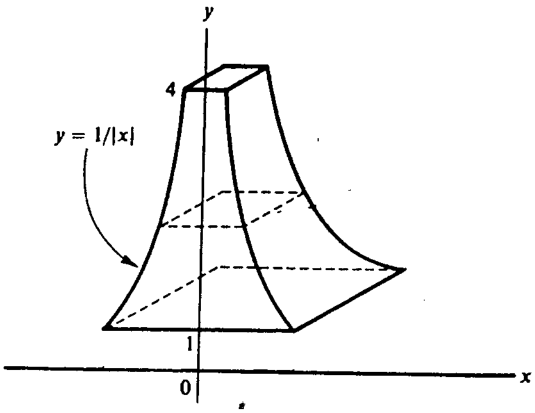
(vi) Show that the point $P\left(2\sec\theta, \sqrt{5}\tan\theta\right)$ lies on the hyperbola and prove that the tangent to the hyperbola at *P* has the equation **2**

$$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1$$

(vii) If the tangent at *P* cuts the asymptotes at *L* and *M*, prove that *LP* = *PM* and the area of triangle *OLM* is independent of the position of *P*. **4**

Question 6. (15 marks) Use a SEPARATE writing booklet. **Marks**

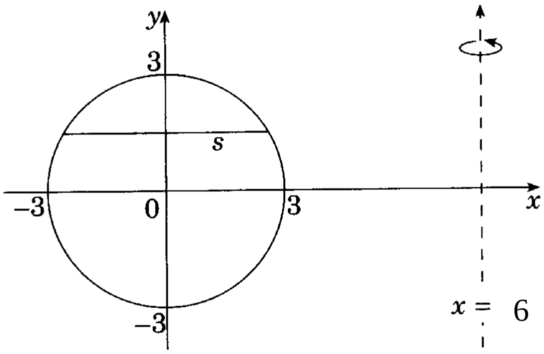
(a) The plan of a steeple is bounded by the curve $y = \frac{1}{|x|}$ and the lines $y = 4$ and $y = 1$.



Each horizontal cross-section is a square.

Find the volume of the steeple. **4**

(b) The circle $x^2 + y^2 = 9$ is rotated about the line $x = 6$ to form a ring.

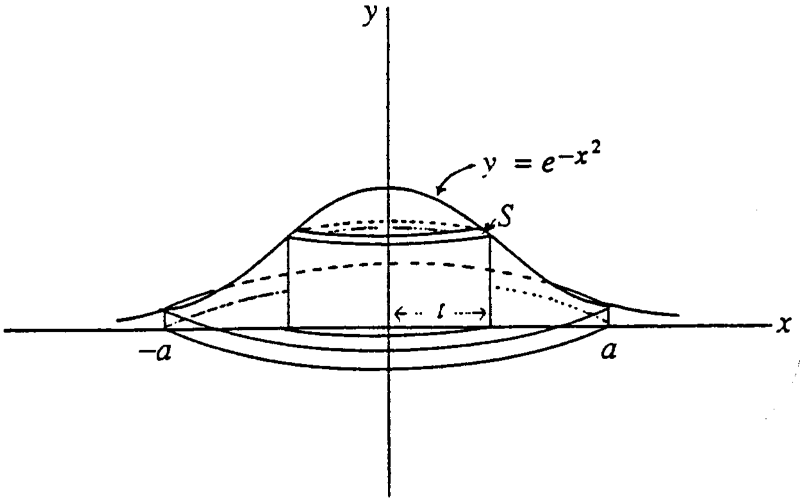


(i) When the circle is rotated, the line segment S at height y sweeps out an annulus. **2**

Find the area of the Annulus.

(ii) Hence find the volume of the ring **3**

(c) The region under the curve $y = e^{-x^2}$ and above the x-axis is rotated about the y axis for $-a \le x \le a$ to form a solid as shown below.



(i) Divide the resulting solid into cylindrical shells S of radius t as shown in the diagram and show each shell S has an approximate volume given by $\delta V = 2\pi t e^{-t^2}\delta t$, where δt is the thickness of the shell. **2**

(ii) Hence calculate the volume of the solid. **2**

(iii) What is the limiting value of the volume of the solid as $a \to \infty$? **2**

Question 7. (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_0^1 (1-x^2)^n \, dx$.

(i) Show by using integration by parts $I_n = \dfrac{2n}{2n+1} I_{n-1}$ for $n = 0, 1, 2, 3, ...$ **3**

(ii) Hence evaluate $\int_0^1 (1 - x^2)^4 dx$ **3**

(b) A special dish is designed by rotating the region bounded by the curve $y = 2\cos x$ $(0 \le x \le 2\pi)$ and the line $y = 2$ through 360^{o} about the y axis.

i) Use the method of cylindrical shells to show that the volume of the dish is given by **3**

$$4\pi \int_0^{2\pi} x(1-\cos x)dx \, .$$

ii) Hence find the volume. **3**

(c) The polynomial $P(x)$ is given by $P(x) = 2x^3 - 9x^2 + 12x - k$, where k is real. **3**

Find the range of values for k for which $P(x) = 0$ has 3 real roots.

Question 8. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use integration by parts to find $\int \sin^{-1} x \ \ dx$. **3**

(b) (i) Use De Moivre's Theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ **3**

(iv) Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots **3**

$$x_1 = \cos\frac{\pi}{12},\ x_2 = -\cos\frac{\pi}{12},\ x_3 = \cos\frac{5\pi}{12},\ x_4 = -\cos\frac{5\pi}{12}$$

(iii) Hence show that $\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$ **2**

(c) $P(x)$ is a polynomial of degree *n* with rational coefficients. **4**

If the leading coefficient is a_0 and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are the roots of $P(x) = 0$ prove that:

$$P'(x) = \frac{P(x)}{x-\alpha_1} + \frac{P(x)}{x-\alpha_2} + \frac{P(x)}{x-\alpha_3} + ...\frac{P(x)}{x-\alpha_n}$$

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Mathematics Extension 2

Solutions

Question 1. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \cos^5 x \sin x \; dx$.

Solution

2	Correct Solution
1	Omitting constant or negative sign

$$= \quad -\frac{1}{6}\cos^6 x \;+\; C$$

(b) Find $\int \frac{2x}{\sqrt{x^2-4}}\, dx$ using the substitution $u = x^2 - 4$.

Solution

$$u = x^2 - 4$$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

$$\int \frac{2x}{\sqrt{x^2-4}}\, dx = \int \frac{2x}{\sqrt{u}}\,\frac{du}{2x}$$

$$= \int \frac{1}{\sqrt{u}}\, du$$

$$= 2\sqrt{u} \;+\; C$$

$$= 2\sqrt{x^2-4} \;+\; C$$

3	Correct Solution
2	Showing $\int \frac{1}{\sqrt{u}}\, du = 2\sqrt{u} \;+\; C$
1	Finding $dx = \frac{du}{2x}$

(c) (i) Express $\frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)}$ as a sum of partial fractions.

Solution

$$\frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{ax+b}{x^2+1}+\frac{cx+d}{x^2+4}$$

2	Correct Solution
1	Finding the expansion or arithmetic error

$$2-x^2 = \left(x^2+4\right)\left(ax+b\right)+\left(x^2+1\right)\left(cx+d\right)$$

$$2-x^2 = ax^3+bx^2+4ax+4b+cx^3+dx^2+cx+d$$

Equating co-efficients & solving simultaneously $a=0, b=1, c=0, d=-2$

$$\frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{1}{x^2+1}-\frac{2}{x^2+4}$$

(ii) Hence show that $\int_0^3 \frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)} = \tan^{-1}\left(\frac{3}{11}\right)$.

Solution

$$\int_0^3 \frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)}dx = \int_0^3 \frac{1}{x^2+1}\,dx - \int_0^3 \frac{2}{x^2+4}\,dx$$

$$= \left[\tan^{-1} x\right]_0^3 - 2\int_0^3 \frac{1}{4+x^2}\,dx$$

$$= \tan^{-1} 3 - 2\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^3$$

$$= \tan^{-1} 3 - \tan^{-1}\frac{3}{2}$$

Let $x = \tan^{-1} 3 \quad y = \tan^{-1}\frac{3}{2}$

$\tan x = 3, \qquad \frac{3}{2} = \tan y$

$$\tan\left(x-y\right) = \frac{\tan x - \tan y}{1+\tan x \tan y}$$

$$= \frac{3-\frac{3}{2}}{1+3.\frac{3}{2}}$$

$$\tan\left(x-y\right) = \frac{3}{11}$$

$$x - y = \tan^{-1}\frac{3}{11}$$

$$\tan^{-1} 3 - \tan^{-1}\frac{3}{2} = \tan^{-1}\frac{3}{11}$$

4	Correct Solution
3	One arithmetic error with correct process
2	Finding $\int_0^3 \frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)}dx = \tan^{-1} 3 - \tan^{-1}\frac{3}{2}$
1	Finding $\left[\tan^{-1} x\right]_0^3 - 2\int_0^3 \frac{1}{4+x^2}\,dx$

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x}$.

Solution

$$t \;=\; \tan\frac{x}{2} \qquad\qquad \int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x} \;=\; \int_0^1 \frac{\frac{2dt}{1+t^2}}{2-\frac{2t}{1+t^2}}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} \qquad\qquad = \int_0^1 \frac{\frac{2dt}{1+t^2}}{\frac{2+2t^2-2t}{1+t^2}}$$

$$\frac{dt}{dx} = \frac{1}{2}\left(1+t^2\right) \qquad\qquad = \int_0^1 \frac{1}{t^2-t+1}\ dt$$

$$dx = \frac{2dt}{1+t^2} \qquad\qquad = \int_0^1 \frac{1}{\left(t^2-t+\frac{1}{4}\right)+\frac{3}{4}}\ dt$$

$$x = \frac{\pi}{2},\ t = \tan\frac{\pi/2}{2} = 1 \qquad\qquad = \int_0^1 \frac{1}{\left(t^2-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\ dt$$

$$x = 0,\ t = \tan\frac{\pi/2}{2} = 1 \qquad\qquad = \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{2\left(t-\frac{1}{2}\right)}{\sqrt{3}}\right]_0^1$$

$$= \frac{2\pi}{\sqrt{3}}\left(\frac{}{3}\right)$$

4	Correct Solution
3	One error with correct process
2	Finding $\int_0^1 \frac{1}{t^2-t+1}\ dt$
1	Finding $dx = \frac{2dt}{1+t^2}$

Question 2. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 1 + i$ and $w = 1 - 2i$. Find in the form $x + iy$,

(i) $z\overline{w}$

Solution

$$z\overline{w} = (1+i)(1+2i)$$

$$= -1 + 3i$$

1	Correct Solution

(ii) $3z + iw$

Solution

$$3z + iw = 3 + 3i + i(1 - 2i)$$

$$= 5 + 4i$$

1	Correct Solution

(iii) $\dfrac{w}{z}$

Solution

$$\frac{w}{z} = \frac{1-2i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{-1}{2} - \frac{3}{2}i$$

1	Correct Solution

(b) Let $\beta = -1 + i$

(i) Express β in modulus-argument form.

Solution

$$|\beta| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} \qquad \arg\beta = \tan^{-1}\left(\frac{-1}{1}\right), \text{ in second quad. } = \frac{3\pi}{4}$$

$$\beta = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

2	Correct Solution
1	Finding either $\lvert\beta\rvert$ or $\arg\beta$

(ii) Express β^4 in modulus-argument form.

Solution

$$\beta^4 = \left[\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)\right]^4$$

1	Correct Solution

$$= 4\left(\cos\left(3\pi\right)+i\sin\left(3\pi\right)\right)$$ by De Moivre's theorem

(iii) Hence evaluate β^{20}

Solution

$$\beta^{20} = \left(\beta^4\right)^5$$

$$= \left(4\left(\cos\left(3\pi\right)+i\sin\left(3\pi\right)\right)\right)^5$$

$$= \left(-4\right)^5$$

$$= -1024$$

1	Correct Solution
0	If student did not interpret HENCE and reapplied De Moivre's Theorem

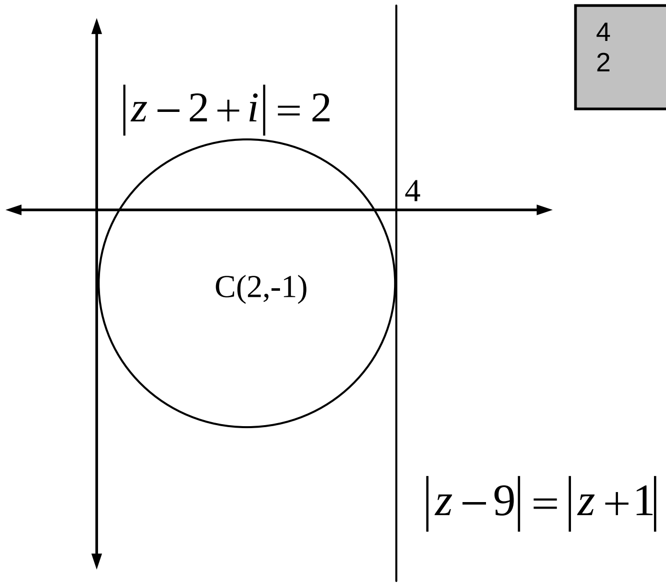
(c) (i) Sketch, on the same Argand diagram, the locus specified by,

1. $\left|z-9\right| = \left|z+1\right|$

2. $\left|z-2+i\right| = 2$

Solution

4	Correct Solution
2	For one correct solution



(ii) Hence write down all the values of z which satisfy simultaneously

$$|z-9|=|z+1| \quad \text{and} \quad |z-2+i|=2$$

Solution

$z=4-i$

1	Correct Solution

(d) Prove $|z_1-z_2|^2+|z_1+z_2|^2=2|z_1|^2+2|z_2|^2$ and interpret this result geometrically.

Solution

Using the property $|w|^2=w\overline{w}$

$$\begin{aligned}
\text{LHS} &= |z_1-z_2|^2+|z_1+z_2|^2\\
&= (z_1-z_2)\left(\overline{z_1-z_2}\right)+(z_1+z_2)\left(\overline{z_1+z_2}\right)\\
&= (z_1-z_2)\left(\overline{z_1}-\overline{z_2}\right)+(z_1+z_2)\left(\overline{z_1}+\overline{z_2}\right)\\
&= z_1\overline{z_1}-z_1\overline{z_2}-z_2\overline{z_1}+z_2\overline{z_2} \;+\; z_1\overline{z_1}+z_1\overline{z_2}+z_2\overline{z_1}+z_2\overline{z_2}\\
&= 2z_1\overline{z_1}+2z_2\overline{z_2}\\
&= 2|z_1|^2+2|z_2|^2\\
&= \text{RHS}
\end{aligned}$$

Geometric interpretation

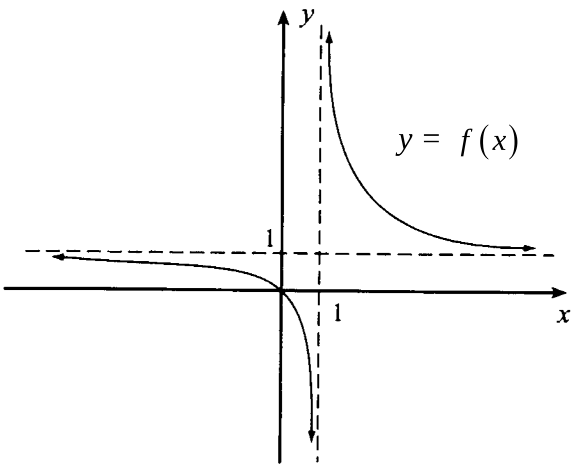
Since z_1-z_2 and z_1+z_2 are diagonals of a parallelogram formed by opposite vertices z_1 and z_2 we can say

The sum of the diagonals squared is equal to 2 times the sum of adjacent sides squared.

3	Correct Solution
2	Showed identity without geometric interpretation.
1	Demonstrating $|w|^2=w\overline{w}$ or $\overline{z_1\pm z_2}=\overline{z_1}\pm\overline{z_2}$

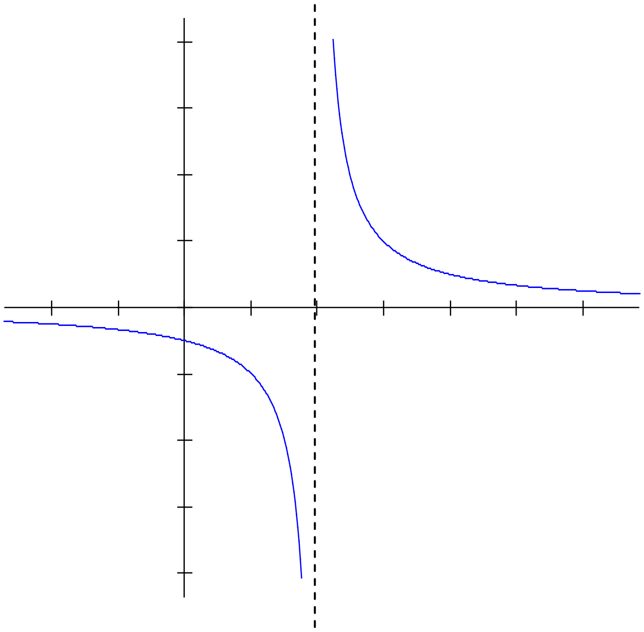
Question 3. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram bellows shows the graph of $y = f(x)$



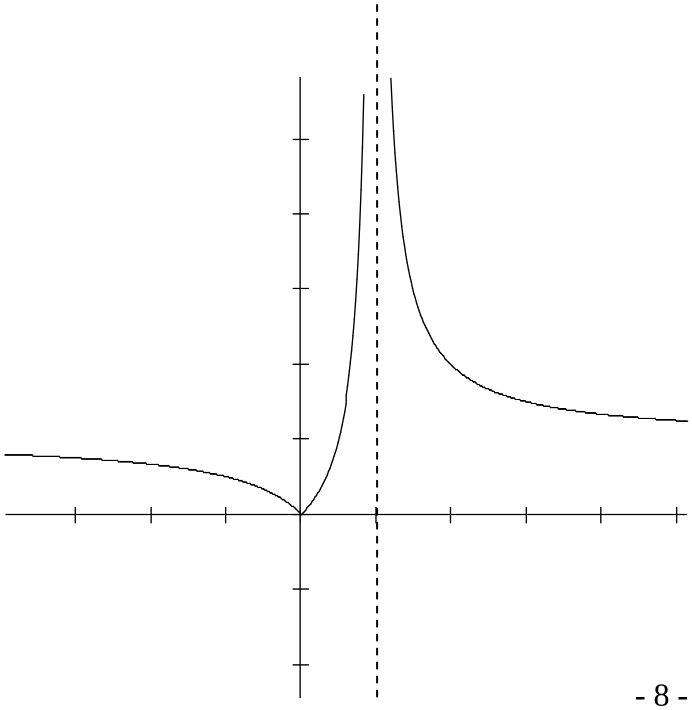
Draw separate one-third page sketches of the graphs

(i) $y = f(x-1)-1$



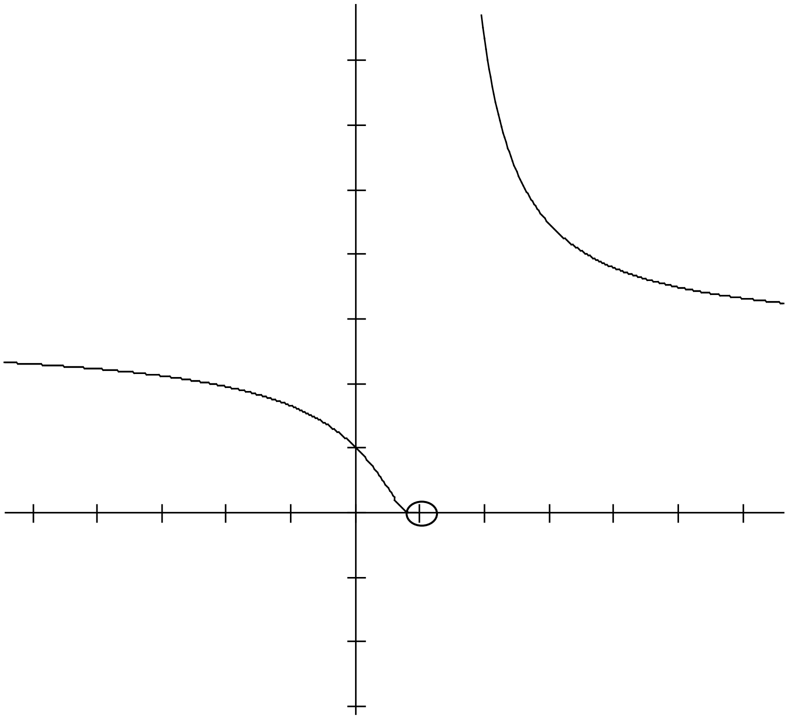
2	Correct Solution
1	For one correct translation only

ii) $y = \left|f(x)\right|$



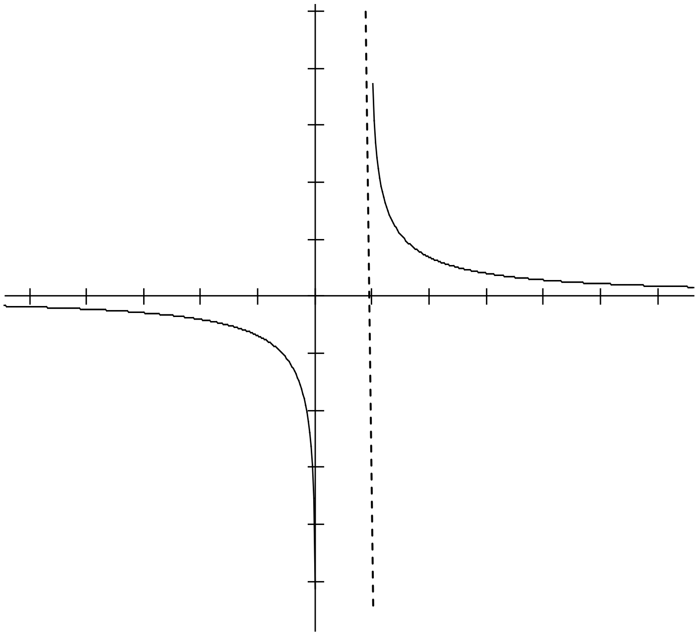
2	Correct Solution
1	Answer sans horizontal asymptote

(iii) $y = e^{f(x)}$



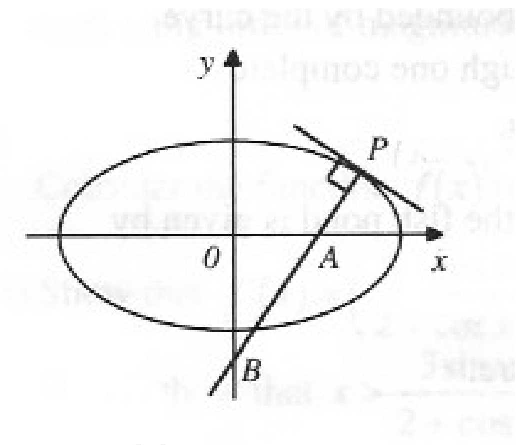
2	Correct Solution
1	For partially correct shape

(iv) $y = \log_e\left(f\left(x\right)\right)$



2	Correct Solution
1	For partially correct shape

(b) $P\left(a\cos\theta, b\sin\theta\right)$, where $0 < \theta < \frac{\pi}{2}$, is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.



The normal at P cuts the x axis at A and the y axis at B.

(i) Show that the normal at P has the equation

$$ax\sin\theta - by\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta$$

Solution

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{2x^2}{a^2} \times \frac{b^2}{2y^2}$$

$$\frac{dy}{dx} = -\frac{b^2}{a^2} \times \frac{x}{y}$$

so the gradient of the tangent at $P\ (x_1, y_1)$ is $-\frac{b^2}{a^2} \times \frac{x_1}{y_1}$

The gradient of the normal is $-\frac{b^2}{a^2}\times\frac{x_1}{y_1}\times m = -1$

$$m = \frac{a^2}{b^2}\times\frac{y_1}{x_1}$$

$$m = \frac{a}{b}\frac{\sin\theta}{\cos\theta}$$

So the equation of the normal is

$$y - b\sin\theta = \frac{a}{b}\frac{\sin\theta}{\cos\theta}(x - a\cos\theta)$$

$$by - b^2\sin\theta = a\frac{\sin\theta}{\cos\theta}(x - a\cos\theta)$$

$$\frac{by}{\sin\theta} - b^2 = \frac{ax}{\cos\theta} - a^2$$

2	Correct Solution
1	Finding gradient of the tangent

$$ax\sin\theta - by\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta$$

(ii) Show that triangle *OAB* has areas $\frac{\left(a^2-b^2\right)^2\sin\theta\cos\theta}{2ab}$

Solution

For point A, $y = 0$

$$ax\sin\theta - b(0)\cos\theta = \left(a^2-b^2\right)\sin\theta\cos\theta$$

$$x = \frac{\left(a^2-b^2\right)\cos\theta}{a}$$

For point B, $x = 0$

$$a(0)\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

$$y = \frac{(a^2 - b^2)\sin\theta}{-b}$$

$$\text{Area AOB} = \frac{1}{2}\times\left|\frac{(a^2 - b^2)\cos\theta}{a}\right|\times\left|\frac{(a^2 - b^2)\sin\theta}{-b}\right|$$

$$= \frac{(a^2 - b^2)^2\sin\theta\cos\theta}{2ab} \qquad \text{since } a > b$$

$$= \frac{(a^2 - b^2)^2\sin\theta\cos\theta}{2ab}\times\frac{2}{2}$$

$$= \frac{(a^2 - b^2)^2\, 2\sin\theta\cos\theta}{4ab}$$

$$= \frac{(a^2 - b^2)^2\sin 2\theta}{4ab}$$

2 Correct Solution
1 Finding *x* and *y* coordinates

(iii) Find the maximum area of the triangle *OAB* and the coordinates of *P* when this maximum occurs.

Solution

$$\frac{dA}{d\theta} = \frac{(a^2 - b^2)^2}{4ab}\cos 2\theta\times 2$$

$$= \frac{(a^2 - b^2)^2\cos 2\theta}{2ab}$$

Finding turning pts

$$\frac{(a^2 - b^2)^2\cos 2\theta}{2ab} = 0$$

Alternatively

$\frac{(a^2 - b^2)^2\sin 2\theta}{4ab}$ is a maximum when $\sin 2\theta = 1$

$$\theta = \frac{\pi}{4}, \qquad \text{since} \quad 0 < \theta < \frac{\pi}{2}$$

$$\left(a^2-b^2\right)^2\cos 2\theta = 0$$

$$\cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

Testing for max.

$$\frac{d^2A}{d\theta^2} = \frac{\left(a^2-b^2\right)^2}{2ab}(-\sin 2\theta)\times 2$$

$$= -\frac{\left(a^2-b^2\right)^2\sin 2\theta}{ab}$$

$$< 0 \quad \text{since } a > b \text{ and } 0 < \theta < \frac{\pi}{2}$$

So max when $\theta = \frac{\pi}{4}$

$$A = \frac{\left(a^2-b^2\right)^2\sin\left(2.\frac{\pi}{4}\right)}{4ab}$$

$$= \frac{\left(a^2-b^2\right)^2}{4ab}$$

3	Correct Solution
2	Omitting 1 answer
1	Finding $\frac{dA}{d\theta} = \frac{\left(a^2-b^2\right)^2\cos 2\theta}{2ab}$

$$P = \left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$$

Question 4. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that α, β and γ are the roots to the equation $x^3 - x^2 + 5x - 3 = 0$, find the equation whose roots are αβ, αγ and βγ

Solution

$$P(x) = x^3 - x^2 + 5x - 3$$

$$\alpha\beta, \beta\gamma, \alpha\beta \quad = \quad \frac{\alpha\beta\gamma}{\gamma}, \frac{\alpha\beta\gamma}{\alpha}, \frac{\alpha\beta\gamma}{\beta}$$

$$= \quad \frac{3}{\gamma}, \frac{3}{\alpha}, \frac{3}{\beta} \qquad \text{since } \alpha\beta\gamma = 3$$

Let $x = \dfrac{3}{\alpha} \Rightarrow \alpha = \dfrac{3}{x}$

$$P(\alpha) = \left(\frac{3}{x}\right)^3 - \left(\frac{3}{x}\right)^2 + 5\left(\frac{3}{x}\right) - 3 \quad = \quad 0$$

$$\frac{27}{x^3} - \frac{9}{x^2} + \frac{15}{x} - 3 \quad = \quad 0$$

$$x^3 - 5x^2 + 3x - 9 \quad = \quad 0$$

3	Correct Solution
2	Substituting $\alpha = \dfrac{3}{x}$
1	Finding $\alpha\beta\gamma = 3$

(b) Let α be the complex root of the polynomial $z^7 = 1$ with the smallest possible argument.

Let θ = $\alpha + \alpha^2 + \alpha^4$ and $\phi = \alpha^3 + \alpha^5 + \alpha^6$

(i) Explain why $\alpha^7 = 1$ and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$

Solution

Let $P(x) = z^7 - 1 = 0$ since α is a root $P(\alpha) = \alpha^7 - 1 = 0 \Rightarrow \alpha^7 = 1$ by remainder theorem

Also $P(x) = z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$

Again since α is a root

$$P(\alpha) = (\alpha - 1)(\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0$$

2	Correct Solution
1	for one correct reason

Since $\alpha \neq 1$ $\qquad (\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0$

(ii) Show θ + ϕ = −1 and θϕ = 2

Hence write a quadratic equation whose roots are θ and ϕ

Solution

$$\theta+\phi = \alpha+\alpha^2+\alpha^4 + \alpha^3+\alpha^5+\alpha^6$$

$$= \alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6$$

= -1 using pt (i)

$$\theta\phi = \left(\alpha+\alpha^2+\alpha^4\right)\left(\alpha^3+\alpha^5+\alpha^6\right)$$

$$= \alpha^4+\alpha^6+\alpha^7+\alpha^5+\alpha^7+\alpha^8+\alpha^7+\alpha^9+\alpha^1$$

$$= \alpha^4+\alpha^6+1+\alpha^5+1+\alpha\alpha^7+1+\alpha^7\alpha^2+\alpha^7\alpha^3$$

$$= \alpha^4+\alpha^6+1+\alpha^5+1+\alpha+1+\alpha^2+\alpha^3$$

$$= \alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6+3$$

= 2 using $\theta+\phi$ = -1

A quadratic whose roots are θ,ϕ easiest quadratic would be monic

$$\theta+\phi = \frac{-b}{a} \qquad \theta\phi = \frac{c}{a}$$

-1 = -*b* 2 = *c* since *a* = 1 with monic polynomial

$$P\left(x\right)=x^2+x+2=0$$

3	Correct Solution
2	Finding 2 correct answers
1	Finding 1 correct answer

(iii) Show that $\theta = -\frac{1}{2}+\frac{i\sqrt{7}}{2}$ and $\phi = -\frac{1}{2}-\frac{i\sqrt{7}}{2}$

Solution

Solving $x^2+x+2=0$

$$x = \frac{-1\pm\sqrt{1^2-4.1.2}}{2.1}$$

$$= \frac{-1\pm\sqrt{-7}}{2}$$

$$= \frac{-1\pm\sqrt{7}i}{2}$$

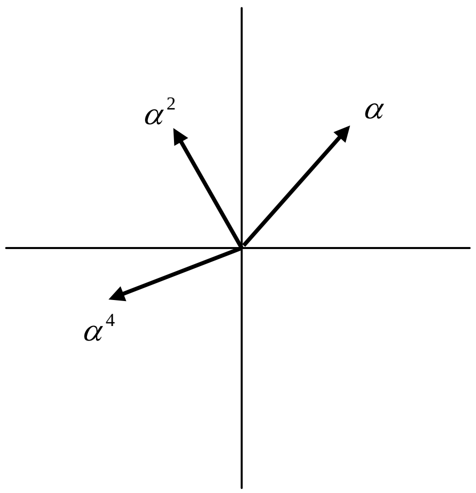
Determining whether $\theta = -\frac{1}{2}+\frac{i\sqrt{7}}{2}$ OR $\phi = -\frac{1}{2}+\frac{i\sqrt{7}}{2}$

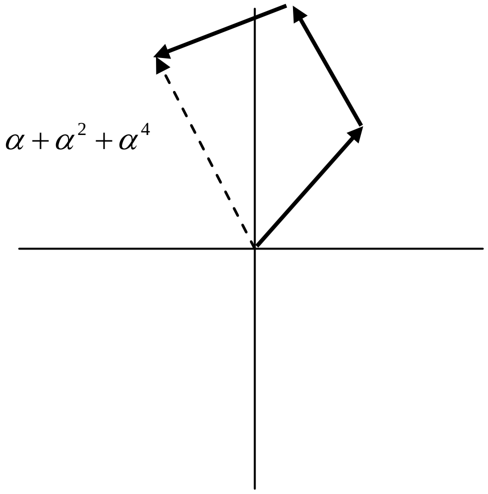
Method 1 We can examine

$\operatorname{Im}\left(\alpha+\alpha^2+\alpha^4\right)=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ [found by solving $z^7=1$ using $cis\frac{2k\pi}{n}$]

> 0 from calculator

Method 2 we can consider addition of vectors $\alpha+\alpha^2+\alpha^4$ [found by solving $z^7=1$ using $cis\frac{2k\pi}{n}$]





Since $\alpha+\alpha^2+\alpha^4$ has a positive argument

$\theta = -\frac{1}{2}+\frac{i\sqrt{7}}{2}$ and $\phi = -\frac{1}{2}-\frac{i\sqrt{7}}{2}$

2	Correct Solution
1	for finding $x = \frac{-1\pm\sqrt{-7}}{2}$

(iv) Write α in modulus argument form and show

$$\cos\frac{4\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}=-\frac{1}{2} \qquad \text{and} \qquad \sin\frac{4\pi}{7}+\sin\frac{2\pi}{7}-\sin\frac{\pi}{7}=\frac{\sqrt{7}}{2}$$

Solution

$\alpha \quad = \quad \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$

$\alpha^2 \quad = \quad \cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}$

$\alpha^4 \quad = \quad \cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}$

$$\theta = \alpha+\alpha^2+\alpha^4$$

$$= \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}+\cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}$$

$$= \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}-\cos\frac{\pi}{7} \quad + \quad i\sin\frac{2\pi}{7}+i\sin\frac{4\pi}{7}-i\sin\frac{\pi}{7}$$

Equating real & imaginary parts (using pt iii)

$$\cos\frac{4\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}=-\frac{1}{2} \qquad \sin\frac{4\pi}{7}+\sin\frac{2\pi}{7}-\sin\frac{\pi}{7}=\frac{\sqrt{7}}{2}$$

2	Correct Solution
1	for finding α

(c) The polynomial $P(z)$ is defined by $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$.

Solution

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of real quadratic factors.

Factors are written in the form $(z - z_1)$

So $z - 2 + i \quad = \quad [z - (2 - i)]$

Since complex roots in a polynomial with real co-efficients occur in complex conjugates another factor is

$$= \quad [z - (2 + i)]$$

Forming a quadratic factor

$$[z-(2-i)][z-(2+i)] \quad = \quad [z-2+i][z-2-i]$$

$$= \quad z^2 - 4z + 5$$

Finding another factor

$$\begin{array}{r} z^2 + 2z + 2 \\ z^2 - 4z + 5\overline{\smash{)}\,z^4 - 2z^3 - z^2 + 2z + 10} \end{array}$$

So product of real quadratic factors

$$P(z) = (z^2 + 2z + 2)(z^2 - 4z + 5)$$

3	Correct Solution
2	for showing long division but with error
1	For finding $z^2 - 4z + 5$

Question 5. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the curve given by $5y - xy = x^2 - x - 2$

(i) Show that the curve has stationary points at 5± 3√2

Solution

$$5\frac{dy}{dx} - x\frac{dy}{dx} - y = 2x - 1$$

$$(5-x)\frac{dy}{dx} = 2x + y - 1$$

$$\frac{dy}{dx} = \frac{2x+y-1}{5-x}$$

Stationary pts at $\frac{dy}{dx} = 0$

$$\frac{2x+y-1}{5-x} = 0$$

$$2x + y - 1 = 0$$

$$y = 1 - 2x$$

Solving for x

$$5(1-2x) - x(1-2x) = (1-2x)^2 - (1-2x) - 2$$

$$5 - 11x + 2x^2 = x^2 - x - 2$$

$$x^2 - 10x + 7 = 0$$

$$x = \frac{10 \pm \sqrt{10^2 - 4.1.7}}{2.1}$$

$$= 5 \pm 3\sqrt{2}$$

2	Correct Solution
1	for finding $\frac{dy}{dx} = \frac{2x+y-1}{5-x}$

(ii) Explain why the curve approaches that of $y=-x-4$ as x → ± ∞

Solution

$$5y-xy = x^2-x-2$$

$$(5-x)y = x^2-x-2$$

$$y = \frac{x^2-x-2}{5-x}$$

$$-x+5\overset{-x-4}{\overline{\big)x^2-x-2}}$$

$$= -x-4+\frac{18}{-x+5}$$

2	Correct Solution
1	for working with partial error

Now $\lim_{x\to\pm\infty} -x-4+\frac{18}{-x+5} = -x-4$

(b) For the hyperbola $\frac{x^2}{4}-\frac{y^2}{5}=1$, find

(i) The eccentricity.

Solution

$$e = \sqrt{\frac{b^2}{a^2}+1}$$

$$= \sqrt{\frac{\left(\sqrt{5}\right)^2}{(2)^2}+1}$$

$$= \frac{3}{2}$$

1	Correct Solution

(ii) The coordinates of the foci.

Solution

Foci $(\pm ae, 0)$

$= \left(\pm 2.\frac{3}{2}, 0\right)$

$= (\pm 3, 0)$

1	Correct Solution

(iii) The equations of the directrices.

Solution

Directrices $x = \pm\frac{a}{e} \Rightarrow \quad x = \pm\frac{4}{3}$

1	Correct Solution

(iv) The equations of the asymptotes.

Solution

$$y = \pm\frac{b}{a}x \quad \Rightarrow \quad y = \pm\frac{\sqrt{5}}{2}x$$

1	Correct Solution

(v) Sketch the hyperbola indicating the foci, the directrices and the asymptotes.

Solution

1	Correct Solution

[Pending]

(vi) Show that the point $P\left(2\sec\theta, \sqrt{5}\tan\theta\right)$ lies on the hyperbola and prove that the tangent to the hyperbola at *P* has the equation

$$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1$$

Solution

Sub *P* into eqn. for hyperbola

$$\begin{aligned}
\text{LHS} &= \frac{\left(2\sec\theta\right)^2}{4} - \frac{\left(\sqrt{5}\tan\theta\right)^2}{5} \\
&= \frac{4\sec^2\theta}{4} - \frac{5\tan^2\theta}{5} \\
&= \sec^2\theta - \tan^2\theta \\
&= \tan^2\theta + 1 - \tan^2\theta \\
&= 1 \\
&= \text{RHS}
\end{aligned}$$

Showing the tangent is $\frac{x\sec\theta}{2}-\frac{y\tan\theta}{\sqrt{5}}=1$

$$\frac{x^2}{4}-\frac{y^2}{5} = 1$$

$$\frac{2x}{4}-\frac{2y}{5}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}=\frac{5x}{4y}$$

At *P* $$\frac{dy}{dx} = \frac{5\sec\theta}{2\sqrt{5}\tan\theta}$$

Equation of a line

$$y-\sqrt{5}\tan\theta = \frac{5\sec\theta}{2\sqrt{5}\tan\theta}\left(x-2\sec\theta\right)$$

$$2\sqrt{5}\tan\theta y-10\tan^2\theta = x5\sec\theta-10\sec^2\theta$$

$$2\sqrt{5}\tan\theta y-10\tan^2\theta = x5\sec\theta-10\tan^2\theta-10$$

$$5\sec\theta x-2\sqrt{5}\tan\theta y = 10$$

$$\frac{x\sec\theta}{2}-\frac{y\sqrt{5}\tan\theta}{5} = 1$$

$$\frac{x\sec\theta}{2}-\frac{y\tan\theta}{\sqrt{5}} = 1 \quad \text{as} \quad \frac{\sqrt{5}}{5}=\frac{1}{\sqrt{5}}$$

2	Correct Solution
1	for showing pt lies on the hyperbola

(vii) If the tangent at *P* cuts the asymptotes at *L* and *M*, prove that *LP* = *PM* and the area of triangle *OLM* is independent of the position of *P*.

Solution

Proving LP = PM

Finding co-ordinates of L & M

Sub asymptote $y = \pm\frac{\sqrt{5}}{2}x$ into the eqn for the tangent

$$\frac{x\sec\theta}{2}-\frac{\left(\frac{\sqrt{5}}{2}x\right)\tan\theta}{\sqrt{5}} = 1 \qquad\qquad \frac{x\sec\theta}{2}-\frac{\left(-\frac{\sqrt{5}}{2}x\right)\tan\theta}{\sqrt{5}} = 1$$

$$\frac{x\sec\theta}{2}-\frac{x\tan\theta}{2} = 1 \qquad\qquad \frac{x\sec\theta}{2}+\frac{x\tan\theta}{2} = 1$$

$$\left(\sec\theta-\tan\theta\right)x = 2 \qquad\qquad \left(\sec\theta+\tan\theta\right)x = 2$$

$$x = \frac{2}{\left(\sec\theta-\tan\theta\right)} \qquad\qquad x = \frac{2}{\left(\sec\theta+\tan\theta\right)}$$

$$\text{and} \quad y = \frac{\sqrt{5}}{\left(\sec\theta-\tan\theta\right)} \qquad\qquad y = \frac{-\sqrt{5}}{\left(\sec\theta+\tan\theta\right)}$$

$$\mathrm{L}\left(\frac{2}{\left(\sec\theta-\tan\theta\right)},\frac{\sqrt{5}}{\left(\sec\theta-\tan\theta\right)}\right) \qquad\qquad \mathrm{M}\left(\frac{2}{\left(\sec\theta+\tan\theta\right)},\frac{-\sqrt{5}}{\left(\sec\theta+\tan\theta\right)}\right)$$

Note finding the distance LP & LM was too troublesome (part marks may be awarded for those who tried) Better to show *P* is the midpoint of LM

$$x = \frac{\frac{2}{\left(\sec\theta-\tan\theta\right)}+\frac{2}{\left(\sec\theta+\tan\theta\right)}}{2} \qquad\qquad y = \frac{\frac{\sqrt{5}}{\left(\sec\theta-\tan\theta\right)}+\frac{-\sqrt{5}}{\left(\sec\theta+\tan\theta\right)}}{2}$$

$$= \frac{1}{\left(\sec\theta-\tan\theta\right)}+\frac{1}{\left(\sec\theta+\tan\theta\right)} \qquad\qquad = \frac{\sqrt{5}}{2}\left(\frac{1}{\left(\sec\theta-\tan\theta\right)}-\frac{1}{\left(\sec\theta+\tan\theta\right)}\right)$$

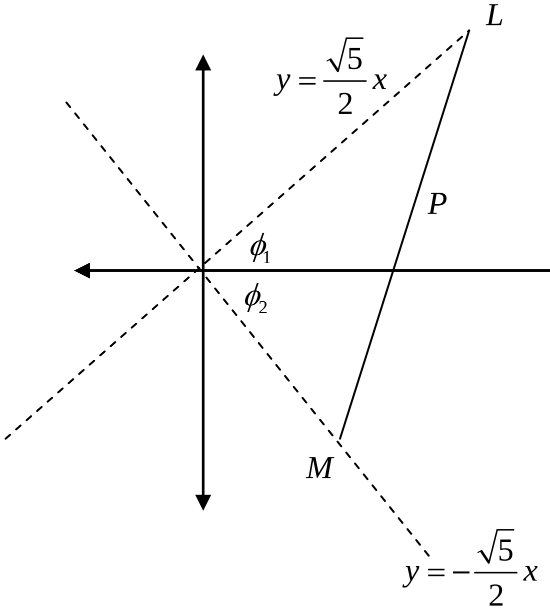
$$= \frac{2\sec\theta}{\sec^2\theta-\tan^2\theta} \qquad\qquad = \frac{\sqrt{5}}{2}\left(\frac{2\tan\theta}{\sec^2\theta-\tan^2\theta}\right)$$

$$= \frac{2\sec\theta}{\tan^2\theta+1-\tan^2\theta} \qquad\qquad = \frac{\sqrt{5}\tan\theta}{\tan^2\theta+1-\tan^2\theta}$$

$$= 2\sec\theta \qquad\qquad = \sqrt{5}\tan\theta$$

Proving *OLM* is independent of *P*

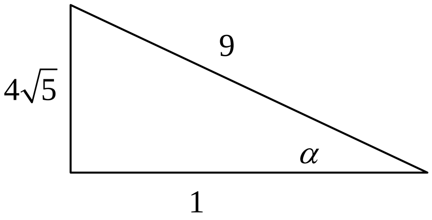
$\tan\phi_1 = \frac{\sqrt{5}}{2}$ $\qquad$ $\tan\phi_2 = \frac{-\sqrt{5}}{2}$



$\phi = \phi_1 - \phi_2$

$$\tan\left(\phi_1 - \phi_2\right) = \frac{\frac{\sqrt{5}}{2} - \frac{-\sqrt{5}}{2}}{1 + \frac{\sqrt{5}}{2}.\left(-\frac{\sqrt{5}}{2}\right)}$$

$$= -4\sqrt{5} \quad \Rightarrow \quad \alpha > \frac{\pi}{2}$$



$\sin\alpha = \frac{4\sqrt{5}}{9}$

$$OL = \sqrt{\left(\frac{2}{\sec\theta - \tan\theta} - 0\right)^2 + \left(\frac{\sqrt{5}}{\sec\theta - \tan\theta} - 0\right)^2} \qquad OM = \sqrt{\left(\frac{2}{\sec\theta - \tan\theta} - 0\right)^2 + \left(\frac{-\sqrt{5}}{\sec\theta - \tan\theta} - 0\right)^2}$$

$$= \sqrt{\frac{9}{\left(\sec\theta - \tan\theta\right)^2}} \qquad = \sqrt{\frac{9}{\left(\sec\theta - \tan\theta\right)^2}}$$

$$= \frac{3}{\sec\theta - \tan\theta} \qquad = \frac{3}{\sec\theta - \tan\theta}$$

$$A = \frac{1}{2}.OL.OM.\sin\alpha$$

$$= \frac{1}{2}.\frac{3}{\sec\theta - \tan\theta}.\frac{3}{\sec\theta - \tan\theta}.\sin\frac{4\sqrt{5}}{9}$$

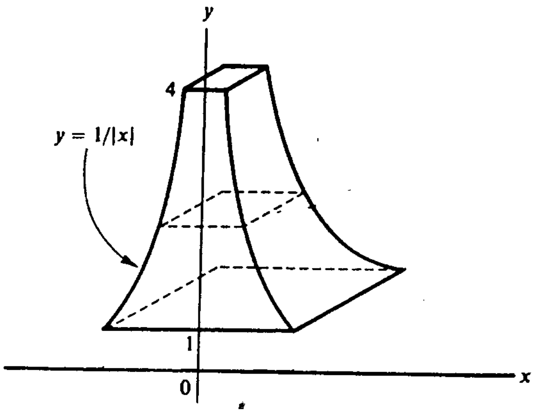
$$= 2\sqrt{5}$$

4	Correct Solution
3	Finding $\sin\alpha$ or *OL* & *OM*
2	Finding the midpoint
1	Finding the co-ordinates of L & M

Which is a constant term and therefore independent of *P*

Question 6. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The plan of a steeple is bounded by the curve $y = \frac{1}{|x|}$ and the lines $y = 4$ and $y = 1$.



Each horizontal cross-section is a square.

Find the volume of the steeple.

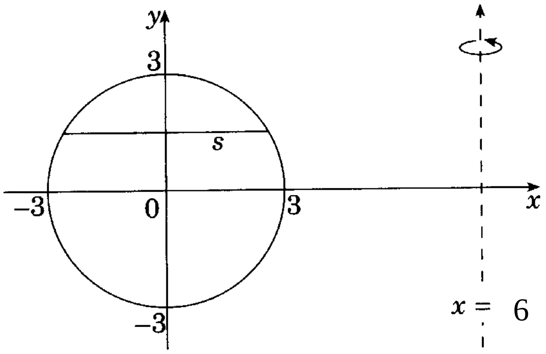
Solution

$$\begin{aligned}\delta V &= (2x)(2x)\delta y \\ &= 4x^2\delta y \\ &= 4.\frac{1}{y^2}.\delta y\end{aligned}$$

$$\begin{aligned}V &= \lim_{\delta y \to 0}\sum_{1}^{4}\frac{4}{y^2}\delta y \\ &= \int_1^4 4y^{-2}\, dy \\ &= \left[\frac{-4}{y}\right]_1^4 \\ &= 3 \text{ units}^3\end{aligned}$$

4	Correct Solution
3	One arithmetic error with correct procedure
2	Finding $\delta V = 4.\frac{1}{y^2}.\delta y$
1	For finding area of a slice

(b) The circle $x^2 + y^2 = 9$ is rotated about the line $x = 6$ to form a ring.



(i) When the circle is rotated, the line segment *S* at height *y* sweeps out an annulus.

Find the area of the Annulus.

Solution

$$A = \pi\left(R^2 - r^2\right)$$

$$= \pi\left([6+x]^2 - [6-x]^2\right)$$

$$= \pi\left([6+x]-[6-x]\right)\left([6+x]+[6-x]\right)$$

$$= 24\pi x$$

2	Correct Solution
1	for showing pt lies on the hyperbola

(ii) Hence find the volume of the ring

Solution

$$\delta V = 24\pi x\delta y$$

$$V = \lim_{\delta y \to 0}\sum_{-3}^{3} 24\pi\left(\sqrt{9-y^2}\right)\delta y$$

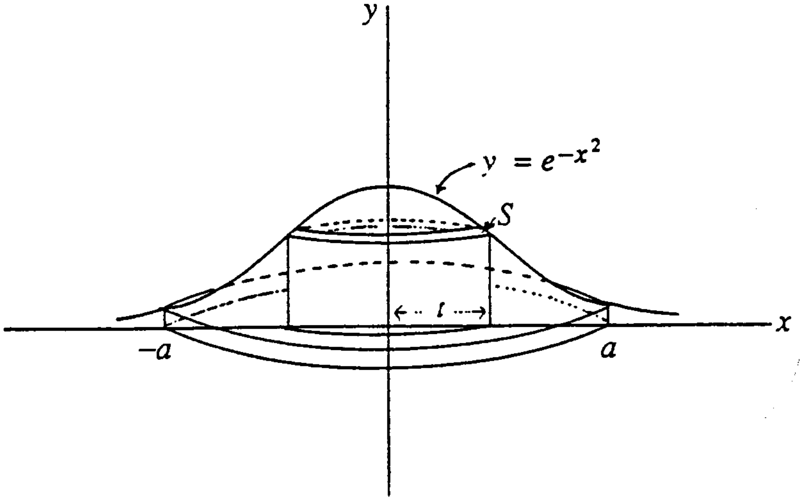
$$= 24\pi\int_{-3}^{3}\sqrt{9-y^2}\ dy$$

$$= 108\pi^2 \quad \text{units}^3$$

3	Correct Solution
2	Showing V = $24\pi\int_{-3}^{3}\sqrt{9-y^2}\ dy$
1	Finding $\delta V = 24\pi x\delta y$

Note $\int_{-3}^{3}\sqrt{9-y^2}\ dy$ is equal to the area of a semi-circle radius 3

(c) The region under the curve $y = e^{-x^2}$ and above the x-axis is rotated about the y axis for $-a \le x \le a$ to form a solid as shown below.



(i) Divide the resulting solid into cylindrical shells S of radius t as shown in the diagram and show each shell S has an approximate volume given by $\delta V = 2\pi t e^{-t^2}\delta t$, where δt is the thickness of the shell.

Solution

$$\delta V = 2\pi r h \;\delta t$$

$$= 2\pi(t)\left(e^{-t^2}\right)\delta t$$

$$= \delta V = 2\pi t e^{-t^2}\delta t$$

2	Correct Solution
1	For partial answer

(ii) Hence calculate the volume of the solid.

Solution

$$V = \lim_{\delta t \to 0}\sum_{0}^{a} 2\pi t e^{-t^2} \;\delta t$$

$$= \pi\int_0^a 2te^{-t^2}\; dt$$

$$= -\pi\int_0^a -2te^{-t^2}\; dt$$

$$= -\pi\left[e^{-t^2}\right]_0^a$$

$$= \pi\left(1-e^{-a^2}\right)$$

2	Correct Solution
1	For some attempt to integrate

(iii) What is the limiting value of the volume of the solid as $a \to \infty$?

Solution

As $a \to \infty, \quad e^{-a^2} \to 0$

So $\lim\limits_{a \to \infty} \pi\left(1 - e^{-a^2}\right) = \pi$

2	Correct Solution
1	For some attempt to integrate

Question 7. (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_0^1 (1-x^2)^n \, dx$.

(i) Show by using integration by parts $I_n = \dfrac{2n}{2n+1} I_{n-1}$ for $n = 0, 1, 2, 3, \ldots$

Solution

$$I_n = \int_0^1 (1-x^2)^n \, dx \quad = \quad \int_0^1 1.(1-x^2)^n \, dx$$

$$= \quad \left[x\left(1-x^2\right)^n\right]_0^1 \; - \; \int_0^1 x\left(n\right)(1-x^2)^{n-1}\left(-2x\right) d$$

$$= \quad 0 \; - \; 2n\int_0^1 -x^2(1-x^2)^{n-1} \; dx$$

$$= \quad -2n\int_0^1 \left(1-x^2-1\right)(1-x^2)^{n-1} \; dx$$

$$= \quad -2n\int_0^1 \left(1-x^2\right)(1-x^2)^{n-1} - 1\left(1-x^2\right)^{n-1} \; dx$$

$$= \quad -2n\int_0^1 \left(1-x^2\right)^n \; + \; 2n\int_0^1 (1-x^2)^{n-1} \; dx$$

$$I_n \quad = \quad -2nI_n \; + \; 2nI_{n-1}$$

$$2nI_n + I_n \quad = \quad 2nI_{n-1}$$

$$I_n \quad = \quad \frac{2n}{2n+1} I_{n-1}$$

3	Correct Solution
2	For finding $-2n\int_0^1 \left(1-x^2\right)^n \; + \; 2n\int_0^1 (1-x^2)^{n-1} \; dx$
1	For finding $0 \; - \; 2n\int_0^1 -x^2(1-x^2)^{n-1} \; dx$

(ii) Hence evaluate $\int_0^1 (1 - x^2)^4 dx$

Solution

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

$$\begin{aligned} I_4 &= \frac{8}{9} I_3 \\ &= \frac{8}{9}\left[\frac{6}{7} I_2\right] \\ &= \frac{8}{9}\left[\frac{6}{7}\right]\left[\frac{4}{5} I_1\right] \\ &= \frac{8}{9}\left[\frac{6}{7}\right]\left[\frac{4}{5}\right]\left[\frac{2}{3} I_0\right] \\ &= \frac{128}{315} \quad \text{as } I_0 = \int_0^1 1 dx = 1 \end{aligned}$$

3	Correct Solution
2	For using recurrence without evaluating last integral
1	For some evidence of correct procedure

(b) A special dish is designed by rotating the region bounded by the curve $y = 2\cos x$ $(0 \le x \le 2\pi)$ and the line $y = 2$ through 360° about the y axis.

i) Use the method of cylindrical shells to show that the volume of the dish is given by

$$4\pi \int_0^{2\pi} x(1 - \cos x) dx .$$

Solution

Each cylindrical shell has height h=2-y and radius x.

$$\delta V = 2\pi r h \, \delta x$$

$$\delta V = 2\pi x \left(2 - y\right) \delta x$$

$$\delta V = 2\pi x \left(2 - 2\cos x\right) \delta x$$

$$\text{Volume of dish} = \lim_{dx \to 0} \sum_{x=0}^{2\pi} 2\pi x \left(2 - 2\cos x\right) \delta x$$

$$= 4\pi \int_0^{2\pi} x \left(1 - \cos x\right) dx$$

3	Correct Solution
2	For showing $\lim_{dx \to 0} \sum_{x=0}^{2\pi} 2\pi x \left(2 - 2\cos x\right) \delta x$
1	For showing $\delta V = 2\pi x \left(2 - y\right) \delta x$

ii) Hence find the volume.

Solution

$$V = 4\pi\int_0^{2\pi} x\left(1-\cos x\right)dx$$

$$V = 4\pi\int_0^{2\pi} x\,dx - 4\pi\int_0^{2\pi} x\cos x\,dx$$

$$V = 4\pi\left[\frac{x^2}{2}\right]_0^{2\pi} - 4\pi\left\{\left[x\sin x\right]_0^{2\pi} - \int_0^{2\pi}\sin x\,dx\right\}$$

$$V = 4\pi\left[\frac{x^2}{2} - x\sin x - \cos x\right]_0^{2\pi}$$

$$V = 4\pi\left[\frac{4\pi^2}{2} - 0 - 1 - \left(0-0-1\right)\right]$$

$$V = 8\pi^3\ units^3$$

3	Correct Solution
2	For showing $4\pi\left[\frac{x^2}{2} - x\sin x - \cos x\right]_0^{2\pi}$
1	For showing $4\pi\left[\frac{x^2}{2}\right]_0^{2\pi} - 4\pi\left\{\left[x\sin x\right]_0^{2\pi} - \int_0^{2\pi}\sin x\,dx\right\}$

(c) The polynomial $P\left(x\right)$ is given by $P\left(x\right) = 2x^3 - 9x^2 + 12x - k$, where k is real. Find the range of values for k for which $P\left(x\right) = 0$ has 3 real roots.

Solution

$P\left(x\right) = 2x^3 - 9x^2 + 12x - k$ will have 3 real roots when $P(x)$ has turning points on either side of the x axis.

Finding turning points:

$$P'\left(x\right) = 6x^2 - 18x + 12$$

$$x^2 - 3x + 2 = 0$$

$$\left(x-2\right)\left(x+1\right) = 0$$

$$x = 2,\ \text{-}1$$

$$y = 4-k,\ 5-k$$

3	Correct Solution
2	For finding the turning points and $y_1y_2 < 0$
1	For finding the turning points

If turning points are on opposite sides $y_1y_2 < 0$

$$\left(4-k\right)\left(5-k\right) < 0$$

$$4 < k < 5$$

Question 8. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use integration by parts to find $\int \sin^{-1} x \ \ dx$.

3	Correct Solution
2	For finding $x\sin^{-1} x - \int x.\frac{1}{\sqrt{1-x^2}}$
1	For finding one part of the IBP e.g. $x\sin^{-1} x$

Solution

$$\int \sin^{-1} x \ \ dx \quad = \quad x\sin^{-1} x - \int x.\frac{1}{\sqrt{1-x^2}}$$

$$= \quad x\sin^{-1} x - \int x\left(1-x^2\right)^{-\frac{1}{2}}$$

$$= \quad x\sin^{-1} x + \sqrt{1-x^2} \ + \ C \qquad \text{as} \quad x\left(1-x^2\right)^{-\frac{1}{2}} = \int -f'(x)\, f(x)$$

(b) (i) Use De Moivre’s Theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

Solution

Let $z = \cos\theta + i\sin\theta$

$$z^4 = \left(\cos\theta + i\sin\theta\right)^4$$

$$= \cos^4\theta + 4\cos^3\theta i\sin\theta + 6\cos^2\theta i^2\sin^2\theta + 4\cos\theta i^3\sin^3\theta + i^4\sin^4\theta$$

$$= \cos^4\theta + 4\cos^3\theta i\sin\theta - 6\cos^2\theta\sin^2\theta - 4\cos\theta i\sin^3\theta + \sin^4\theta$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \quad + \quad 4\cos^3\theta i\sin\theta - 4\cos\theta i\sin^3\theta$$

Also by De Moivre’s theorem

$$z^4 = \left(\cos\theta + i\sin\theta\right)^4$$

$$= \ \cos 4\theta + i\sin 4\theta$$

3	Correct Solution
2	For finding $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
1	For finding $z^4 = \cos 4\theta + i\sin 4\theta$

Equating real parts

$$\cos 4\theta = \quad \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= \quad \cos^4\theta - 6\cos^2\theta\left(1-\cos^2\theta\right) + \left(\sin^2\theta\right)\left(\sin^2\theta\right)$$

$$= \quad \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + \left(1-\cos^2\theta\right)\left(1-\cos^2\theta\right)$$

$$= \quad 8\cos^4\theta - 8\cos^2\theta + 1$$

(iv) Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots

$$x_1 = \cos\frac{\pi}{12},\ x_2 = -\cos\frac{\pi}{12},\ x_3 = \cos\frac{5\pi}{12},\ x_4 = -\cos\frac{5\pi}{12}$$

Solution

$16x^4 - 16x^2 + 1 = 0$

Let $x = \cos\theta$

$16\cos^4\theta - 16\cos^2\theta + 1 = 0$

$2\left(8\cos^4\theta - 16\cos^2\theta + 1\right) - 1 = 0$

3	Correct Solution
2	For finding $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \ldots$
1	For finding $2\left(8\cos^4\theta - 16\cos^2\theta + 1\right) - 1 = 0$

$$2\cos 4\theta - 1 = 0$$

$$\cos 4\theta = \frac{1}{2}$$

$$4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \ldots$$

Now $x = \cos\theta$

$$x = \cos\frac{\pi}{12}, \qquad x = \cos\frac{5\pi}{12}, \qquad x = \cos\frac{7\pi}{12} = -\cos\frac{5\pi}{12}, \qquad x = \cos\frac{11\pi}{12} = -\cos\frac{\pi}{12}$$

(iii) Hence show that $\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$

Solution

Solving $16x^4 - 16x^2 + 1 = 0$ by using quadratics

Let $m = x^2$ $16m^2 - 16m^2 + 1 = 0$

$$m = \frac{16 \pm \sqrt{(-16)^2 - 4.16.1}}{2.16}$$

$$m=\frac{16\pm 8\sqrt{3}}{32}$$

$$m=\frac{2\pm\sqrt{3}}{4}$$

$$x^2=\frac{2\pm\sqrt{3}}{4}$$

$$x=\pm\sqrt{\frac{2\pm\sqrt{3}}{4}}$$

Examining the positive roots as $\cos\frac{\pi}{12}>0$

$$=\frac{\sqrt{2+\sqrt{3}}}{2}\quad\text{or}\quad\frac{\sqrt{2-\sqrt{3}}}{2}$$

Since $\frac{\sqrt{2+\sqrt{3}}}{2}>\frac{\sqrt{2-\sqrt{3}}}{2}$ and $\cos\frac{\pi}{12}>\cos\frac{5\pi}{12}$

$$\cos\frac{\pi}{12}=\frac{\sqrt{2+\sqrt{3}}}{2}$$

(c) $P(x)$ is a polynomial of degree *n* with rational coefficients.

If the leading coefficient is a_0 and $\alpha_1,\alpha_2,\alpha_3,...,\alpha_n$ are the roots of $P(x)=0$ prove that:

$$P'(x)=\frac{P(x)}{x-\alpha_1}+\frac{P(x)}{x-\alpha_2}+\frac{P(x)}{x-\alpha_3}+...\frac{P(x)}{x-\alpha_n}$$

Solution

$$P(x)=a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)...(x-\alpha_n)$$

$$\log_e P(x)=\log_e a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)...(x-\alpha_n)$$

$$\log_e P(x)=\log_e a_0+\log_e(x-\alpha_1)+\log_e(x-\alpha_2)+\log_e(x-\alpha_3)+...+\log_e a_0(x-\alpha_n)$$

$$\frac{P'(x)}{P(x)}=0+\frac{1}{(x-\alpha_1)}+\frac{1}{(x-\alpha_2)}+\frac{1}{(x-\alpha_3)}+...+\frac{1}{(x-\alpha_n)}$$

$$P'(x)=\frac{P(x)}{x-\alpha_1}+\frac{P(x)}{x-\alpha_2}+\frac{P(x)}{x-\alpha_3}+...\frac{P(x)}{x-\alpha_n}$$

4	Correct Solution
3	For finding
2	For finding ?
1	For finding $P(x)=a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)...(x-\alpha_n)$